Model equations for two-dimensional quasipatterns

Hanns Walter Müller
Ecole Normale Supérieure de Lyon, 46 Allée d'Italie, 69364 Lyon, France
and Institut für Theoretische Physik, Universität des Saarlandes Bau 38, 66041 Saarbrücken, Germany
(Received 18 October 1993)

Two-dimensional model equations, invariant under the symmetry operations of an infinite horizontal plane, can undergo a cellular instability to "quasipatterns" of eight- or 12-fold orientational order. Two different nonlinear selection mechanisms are discussed and related to recent experiments.

PACS number(s): 47.54.+r, 61.44.+p, 47.20.Ky

The appearance of quasicrystalline order has first been observed in solid-state physics [1]. Only recently have planar patterns with an eight- or 12-fold orientational order (in the following denoted as "quasipatterns" due to their quasiperiodic long-range translational order) been found in a dissipative hydrodynamic system [2,3]. This was a surprising result since the tilings observed so far had been either lines, squares, or hexagons. A common property of these systems is a continuous degeneracy associated with a rotational symmetry in the horizontal plane. As far as linear dynamics is concerned, all Fourier modes $e^{i\mathbf{k}\cdot\mathbf{r}}$ with $|\mathbf{k}|=k_c$ are equally amplified, but only a few of them survive and saturate due to nonlinear interactions. The experimental quasipatterns [3] are independent of the lateral boundaries and are thus likely to be the result of nonlinear bulk effects. In this paper we present two-dimensional model equations with a relaxational dynamics towards eight- and 12-fold quasipatterns. The nonlinear mechanisms responsible for the pattern selection might give an insight into the physics prevailing in experiments.

The starting point is a two-dimensional Swift-Hohenberg (SH) -type equation [4] for a real scalar field $u(x,y,t)$

$$\partial_t u=\epsilon u-(\nabla^2+1)^2 u-c(u)\ , \tag{1}$$

where $c(u)$ is a general cubic nonlinearity and $\nabla=(\partial_x,\partial_y)$. The linear operator in (1) gives a stationary instability at a finite wave number $k_c=1$. The direction of $\mathbf{k}=(k_x,k_y)$ is infinitely degenerate, reflecting the continuous rotational symmetry in the x-y plane. In what follows we study the effect of various $c(u)$'s on the pattern selection. The only restrictions upon $c(u)$ are the requirements of space translation, reflection, and rotational invariance as well as saturation of the instability. In this sense the simplest $c(u)$ is $\frac{1}{3}u^3$ [5] by which Eq. (1) becomes the original (SH)-equation producing a pattern of lines [4]. A second known $c(u)$ derived by Chapman and Proctor (CP) [6] is $\frac{1}{3}\nabla\cdot[\nabla u(\nabla u)^2]$. It drives a square tesselation rather than lines. Equation (1) with the SH or the CP nonlinearity is derivable from the Lyapunov functional $\mathcal{L}=(1/2)\int dx\int dy\{-\epsilon u^2+[(\nabla^2+1)u]^2+l(u)\}$, with the positive-definite quartic contribution $l(u)=\frac{1}{6}u^4$ for SH and $\frac{1}{6}(\nabla u)^4$ for CP. The inequality $d\mathcal{L}/dt=-\int dx\int dy(\partial u/\partial t)^2\leq 0$ guarantees a relaxational dynamics towards the state of minimum free energy. Expanding the solution in N Fourier modes $u=\sum_{n=1}^{N}A_n e^{i\mathbf{k}_n\cdot\mathbf{r}}+\text{c.c.}$, where each $\mathbf{k}_n$ lies on the critical unit circle, the temporal evolution of the mode amplitudes $\{A_n\}$ is governed by a set of coupled Landau equations

$$\partial_t A_n=\epsilon A_n-\sum_{m=1}^{N}\beta(\theta_{nm})|A_m|^2 A_n\ ,\quad n=1,\dots,N\ . \tag{2}$$

These are gradient equations with $\partial_t A_n=-\partial F/\partial A_n^*$, where the free energy is

$$F=-\epsilon\sum_{i=1}^{N}|A_i|^2+\tfrac{1}{2}\sum_{i,j=1}^{N}\beta(\theta_{ij})|A_i|^2|A_j|^2\ . \tag{3}$$

The coupling function $\beta(\theta)$, evaluated at the angles θ_{ij} between $\mathbf{k}_i$ and $\mathbf{k}_j$, is characteristic for the $c(u)$ under consideration. It determines which pattern is to be selected. Reflection and rotational symmetry imposed on $c(u)$ imply $\beta(\theta)=\beta(\pi-\theta)=\beta(-\theta)$. Since the number of resonant cubic interactions between two different critical modes is twice the number of self-interactions [7], the coupling function $\beta(\theta)$ is discontinuous at $\theta=0$ and $\theta=\pi$ with $\beta(\theta\to 0)=\beta(\theta\to\pi)=2\beta(0)$. This fact has not been addressed by Refs. [8,9] although it has important consequences for the pattern selection process.

Equation (2) allows equal amplitude fixed point solutions of the form $|A_n|=A$, $|\mathbf{k}_n|=k_c\equiv 1$ and $\sphericalangle(\mathbf{k}_n,\mathbf{k}_m)=|n-m|\pi/N$, with $n,m=1,\dots,N$. They describe regular periodic ($N\leq 3$) or quasiperiodic ($N>3$) patterns of $2N$-fold orientational order (e.g., $N=1$ lines; $N=2$, squares; $N=3$, hexagons or triangles, depending on the relative phases of the A_n; $N=4$, octagonal quasipatterns; ...). The corresponding extrema of the free energy are determined by averaging over the participating matrix elements [7]

$$F_N=-\tfrac{1}{2}\epsilon^2\left[\frac{1}{N}\sum_{n=0}^{N-1}\beta(n\pi/N)\right]^{-1}\ . \tag{4}$$

From this expression the leading role of the coupling

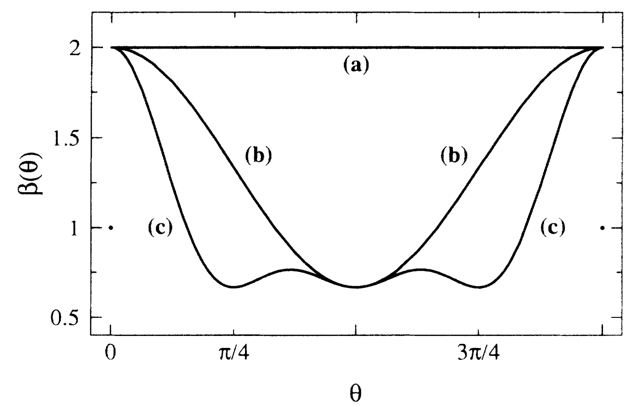

FIG. 1. Coupling function $\beta(\theta)=P(\cos^2(\theta))$ for the cubic nonlinearity: (a) $\frac{1}{3}u^3$ where $P(x)=2$; (b) $-\frac{1}{3}\nabla\cdot[\nabla u(\nabla u)^2]$ where $P(x)=\frac{2}{3}(1+2x)$; (c) for Eq. (6) where $P(x)=\frac{1}{3}(2+4x-16x^2+16x^3)$. Note that $\beta(\theta)$ is discontinuous at $\theta=0$, and $\theta=\pi$ with $\beta(0)=\beta(\pi)=1$.

TABLE I. Coupling function $\beta(\theta)=P(\cos^2(\theta))$ for the first cubic nonlinearities $u\nabla^{2n}(u)^2$. To derive $P(x)$ the expansion $u=\sum_n A_n e^{i\mathbf{k}_n\cdot\mathbf{r}}$+c.c. with $|\mathbf{k}_n|=k_c$ has been used.

$c(u)$	$P(x)$
u^3	6
$u\nabla^2(u^2)$	$8k_c^2$
$u\nabla^4(u^2)$	$8k_c^4(1+3x)$
$u\nabla^6(u^2)$	$8k_c^6(1+15x)$
$u\nabla^8(u^2)$	$8k_c^8(1+28x+35x^2)$
$u\nabla^{10}(u^2)$	$8k_c^{10}(1+45x+210x^2)$
$u\nabla^{12}(u^2)$	$8k_c^{12}(1+66x+495x^2+462x^3)$

function $\beta(\theta)$ becomes evident: Figure 1(a) shows $\beta(\theta)$ belonging to the $\frac{1}{3}u^3$ nonlinearity. Due to the discontinuity at $\theta=0,\pi$ one gets $F_1\leq F_N$ for all N, i.e., the pattern of lowest free energy consists of rolls. For the CP nonlinearity the coupling function is depicted in Fig. 1(b). Here, the resulting relation $F_2\leq F_N$ (for all N) ensures the preference of squares. Obviously, by broadening the minimum of $\beta(\theta)$ [see Fig. 1(c)] one can, e.g., achieve $F_4\leq F_N$, thus stabilizing octagonal quasipatterns. Likewise, patterns with arbitrary orientational order— the "turbulent crystals" of Ref. [7]—can be stabilized.

The idea of the following arguments is to design $\beta(\theta)$, by an appropriate choice of $c(u)$. For any $c(u)$ compatible with the imposed symmetry conditions the coupling function is of the form

$$\beta(\theta)=\begin{cases}\frac{1}{2}P(1) & \text{for } \theta=0,\pi\\ P(\cos^2(\theta)) & \text{for } 0<\theta<\pi\ ,\end{cases} \tag{5}$$

where $P(x)$ is a polynomial in x. The higher its degree, the greater the flexibility in designing $\beta(\theta)$. Since the repeated application of ∇^2 upon u^2 generates powers of $\cos^2(\theta)$ in $\beta(\theta)$, we combine $c(u)$ of the form $u\nabla^{2n}(u^2)$. They derive from positive definite functionals $\int dx\int dy[\sum_n a_n\nabla^{2n}(u^2)]^2$. The present choice of $c(u)$ should be considered as a working example; it is not imperative. By inspection of Eq. (4) it can be shown that $P(x)$ must be of at least third order to achieve $F_4\leq F_N$ (for all N), i.e., to meet the stability condition for eightfold quasipatterns [10]. For this purpose a nonlinear combination of space derivatives up to 12th is necessary (see Table I). Figure 1(c) shows an appropriate coupling function $\beta(\theta)$, which results from

$$c(u)=0.0627u^3+0.0302u\nabla^4(u^2)+u[0.1162\nabla^4-0.0380\nabla^6]^2u^2\ . \tag{6}$$

The right-hand side of Eq. (6) derives from a positive-definite quartic contribution $l(u)$ to the Lyapunov functional. Consequently, the dynamics of Eq. (1) with (6) is relaxational towards eightfold quasipatterns.

The method presented here is constructive and can be used to design partial differential equations (PDE's) for periodic or quasiperiodic patterns of arbitrary orientational order. The analysis shows how a selection mechanism governed by Eq. (2) and an appropriate $\beta(\theta)$ can drive forwards bifurcating two-dimensional quasipatterns. The mechanism contends with a *single* set of linearly unstable wave vectors $|\mathbf{k}|=k_c$. Interactions with a *second* set of (almost) critical modes—recently suggested [9] as a possible stabilizing mechanism for quasipatterns—are not required.

The extreme high-order derivative coupling (6) of the present model seems to be unphysical, since realistic planar pattern-forming systems (e.g., Rayleigh-Bérnard convection or the standing surface patterns in the Faraday instability) generically exhibit simple nonlinearities. On the other hand, in real (i.e., three-dimensional) systems the vertical space direction enters the problem and the horizontal boundary conditions give rise to very complicated dependencies of $\beta(\theta)$ in the amplitude equation (see, e.g., Ref. [11]). In some systems these coupling functions might be complicated enough to stabilize quasipatterns. A possible candidate is the Faraday experiment of Christiansen, Alstrom, and Levinsen [2]. Recently they observed standing surface patterns of eightfold orientational order. Since the structure is in *subharmonic* resonance with the forcing, quadratic terms in the corresponding amplitude equation cannot appear and Eq. (2) applies. The eightfold quasipattern is likely to be the result of an appropriately shaped coupling function $\beta(\theta)$. Another Faraday experiment [12] has been performed with a forcing composed of two frequencies. As a function of their relative amplitudes r and their phase shift ϕ one either observes subharmonic squares or subharmonic hexagons and/or triangles at the onset of the instability. These patterns can also be understood by Eq. (2), since $\beta(\theta)$ parametrically depends on r and ϕ.

The rest of this paper is devoted to a model that is based on another stabilization mechanism for quasicrystals. The idea has been introduced by Mermin and Trojan [9] and generalized by Newell and Pomeau [7]. The point is that a quadratic nonlinearity mediates a triad interaction between two critical sets of modes. Two wave vectors $|\mathbf{k}_1|=|\mathbf{k}_2|=k_c$ and a third one $|\mathbf{q}|=q_c$ form an isosceles triangle, where the angle δ between $\mathbf{k}_1$ and $\mathbf{k}_2$ is tunable by the ratio q_c/k_c (for an octagonal quasipattern this angle must be tuned to $\delta=\pi/4$).

To introduce a second wave number we consider the following pair of quadratically coupled CP-like equations, which become unstable at $k_c=1$ and $q_c\neq 1$, respectively:

$$\partial_t u_1=\epsilon u_1-(\nabla^2+1)^2 u_1+\tfrac{1}{3}\nabla\cdot[\nabla u_1(\nabla u_1)^2] + \tfrac{1}{2}\gamma_1 u_1 u_2 , \quad (7a)$$

$$\partial_t u_2=\mu u_2-(\nabla^2+q_c^2)^2 u_2+\frac{1}{3q_c^4}\nabla\cdot[\nabla u_2(\nabla u_2)^2] + \tfrac{1}{2}\gamma_2 u_1^2 . \quad (7b)$$

The quantities $u_1(x,y,t)$ and $u_2(x,y,t)$ model the order-parameter fields in a bicritical system, where ϵ and μ measure the distance to the respective stability thresholds. The fields are coupled by a quadratic interaction. The model described by Eq. (7) is motivated by the experiment of Edwards and Fauve [3]: In a Faraday setup with a two-frequency forcing ($\omega_1:\omega_2=4:5$) quasipatterns arise close to a bicritical situation in which the surface structures associated with the 5ω and 4ω frequency compete. If γ_1 and γ_2 have the same sign, Eqs. (7) are potential and the dynamics will be relaxational. In the following we put $\gamma_1=\gamma_2=\gamma$ for simplicity. For $\gamma=0$ (coupling turned off) the CP nonlinearities in (7) drive supercritical square patterns in u_1 and u_2. As soon as $\gamma>0$ (the case $\gamma<0$ is recovered by $u_2\to -u_2$) a new class of fixed points can take advantage of the quadratic coupling and the system undertakes a hysteric backwards bifurcation. In order to enforce eightfold quasipatterns the vertex angle between two adjacent wave vectors must be tuned to $\pi/4$. Consequently, the value of q_c must be chosen $|\mathbf{k}_i\pm\mathbf{k}_{i+1}|=(2\pm\sqrt{2})^{1/2}$. To compute the weakly nonlinear solutions of (7) we Fourier expand $u_1=\sum_{n=1}^4 A_n e^{i\mathbf{k}_n\cdot\mathbf{r}}+\text{c.c.}$ with $|\mathbf{k}_n|=1$, $\mathbf{k}_n\cdot\mathbf{k}_{n+1}=\cos(\pi/4)$ and $u_2=\sum_{n=1}^4 B_n e^{i\mathbf{q}_n\cdot\mathbf{r}}+\text{c.c.}$ with $|\mathbf{q}_n|=q_c$, $\mathbf{q}_n\cdot\mathbf{q}_{n+1}=q_c^2\cos(\pi/4)$. The resulting coupled Landau equations for the amplitudes $\{A_n\},\{B_n\}$ derive from the free energy

$$\begin{aligned}F=&-\sum_{i=1}^{4}(\epsilon|A_i|^2+\mu|B_i|^2)\\&+\tfrac{1}{2}\sum_{i,j=1}^{4}\beta(\theta_{ij})(|A_i|^2|A_j|^2+|B_i|^2|B_j|^2)\\&-\gamma(B_1A_2^*A_1^*+B_2A_3^*A_2^*+B_3A_4^*A_3^*\\&+B_4A_1A_4^*+\text{c.c.}) .\end{aligned} \quad (8)$$

The coupling function $\beta(\theta)$ is that of Fig. 1(b). To derive (8) we have assumed $\gamma\ll 1$, which allows one to neglect the γ^2 corrections in $\beta(\theta)$. At this point a remark concerning the work of Newell and Pomeau [7] is in order. They consider the $\{B_n\}$ as "passive" (or damped) modes slaved by the "active" (or supercritical) A modes. Thus, $c(u)$'s in Eq. (7b) are not required in their model, and the whole dynamics takes place on the center manifold $B_n(\{A_m\})$. The corrections to $\beta(\theta)$ arising from the quadratic interactions are essential in their model, as they generate the necessary deformations of the coupling function to favor *supercritical* quasipatterns. In the present model, however, both sets of modes $\{A_n\}$ and $\{B_n\}$ are supposed to be close to threshold, i.e., they both are active and the dynamics takes place in the whole $\{A_n\}-\{B_m\}$ phase space. From $\partial F/\partial A_n^*=0=\partial F/\partial B_m^*$ one can construct four backwards bifurcating fixed-point solutions that take advantage of the quadratic cross coupling. We denote them by the indices $(4\text{-}2)^\pm$ and $(8\text{-}8)^\pm$. Within the brackets the first (second) index counts the number of equally saturated A modes (B modes). The index pair (8-8) represents an eightfold quasipattern in u_1 as well as u_2. The quadratic interaction enforces that the two patterns are twisted by an angle of $\pi/8$ relative to each other. The notation (4-2) represents a periodic pattern with rhombic unit cell in u_1 (vertex angle $\pi/4$) combined with a simple line structure in u_2. The signs + and − denote whether the total phase in a triad interaction adds to an angle of 0 or π, respectively. A linear stability analysis of these nonlinear solutions can be carried out analytically in the case $\epsilon=\mu=0$ (recall that the solutions are hysteric with finite amplitude at threshold). One finds that both $4\text{-}2^+$ and $8\text{-}8^+$ are stable, i.e., minima of the free energy. Using (8), note that the comparison of the corresponding values of F,

$$F_{8\text{-}8+}\equiv-(\tfrac{6}{13})^3\gamma^4<-\tfrac{1}{2}(\tfrac{3}{7})^2\gamma^4\equiv F_{4\text{-}2+} , \quad (9)$$

proves that the eightfold quasipattern fixed point is preferred. Other nonlinear solutions, which cannot take advantage of the quadratic coupling, do not compete close to the bicriticality since their free energies disappear according to $F=O(\epsilon^2,\mu^2)$. A numerical check of the stability prediction (9) has been performed by integrating the ordinary Landau system $\partial_t A_n=\partial F/\partial A_n^*$, $\partial_t B_n=\partial F/B_n^*$ at $\epsilon=\mu=0$ and $\gamma=0.1$. Starting from random initial conditions for the A_n, B_n the long-time solution always saturates in the eightfold quasipattern.

Following the same construction principle we arrive at a model system for 12-fold quasipatterns,

$$\partial_t u_1=\epsilon u_1-(\nabla^2+1)^2 u_1+\alpha_1 u_1^2-u_1^3+\gamma u_1 u_2 , \quad (10a)$$

$$\partial_t u_2=\mu u_2-(\nabla^2+q_c^2)^2 u_2+\alpha_2 u_2^2-u_2^3+\gamma u_1^2 . \quad (10b)$$

In the absence of cross coupling ($\gamma=0$) u_1 and u_2 saturate in hexagonal patterns of wave vector $k_c=1$ and $k_c=q_c$, respectively. If $\gamma\neq 0$ the value of q_c must be tuned to $\sqrt{2}$ or $(2\pm\sqrt{3})^{1/2}$ in order to drive the 12-fold orientational order with vertex angles of $\pi/2$ or $\pi/6$, respectively. Given a hexagonal solution u_2 of wave number q_c, Eq. (10a) can be regarded as a single PDE with space-dependent forcing $(\epsilon+\gamma u_2)u_1$. That way, 12-fold quasipatterns have been found by Frisch [13] in a numerical simulation. We solved the coupled system [(10a) and (10b)] numerically by a spectral code on a 64×64 wave-number grid with periodic boundary conditions. For $\epsilon=\mu=0.01$, $\alpha_1=\alpha_2=\gamma=0.2$ we observed 12-fold quasipatterns, which did not depend very sensitively on the parameter values chosen. Figure 2 presents shadow graphs of the computed patterns together with intensity plots of the associated power spectra.

The foregoing discussion presents models for forwards and slightly backwards bifurcating quasipatterns. The

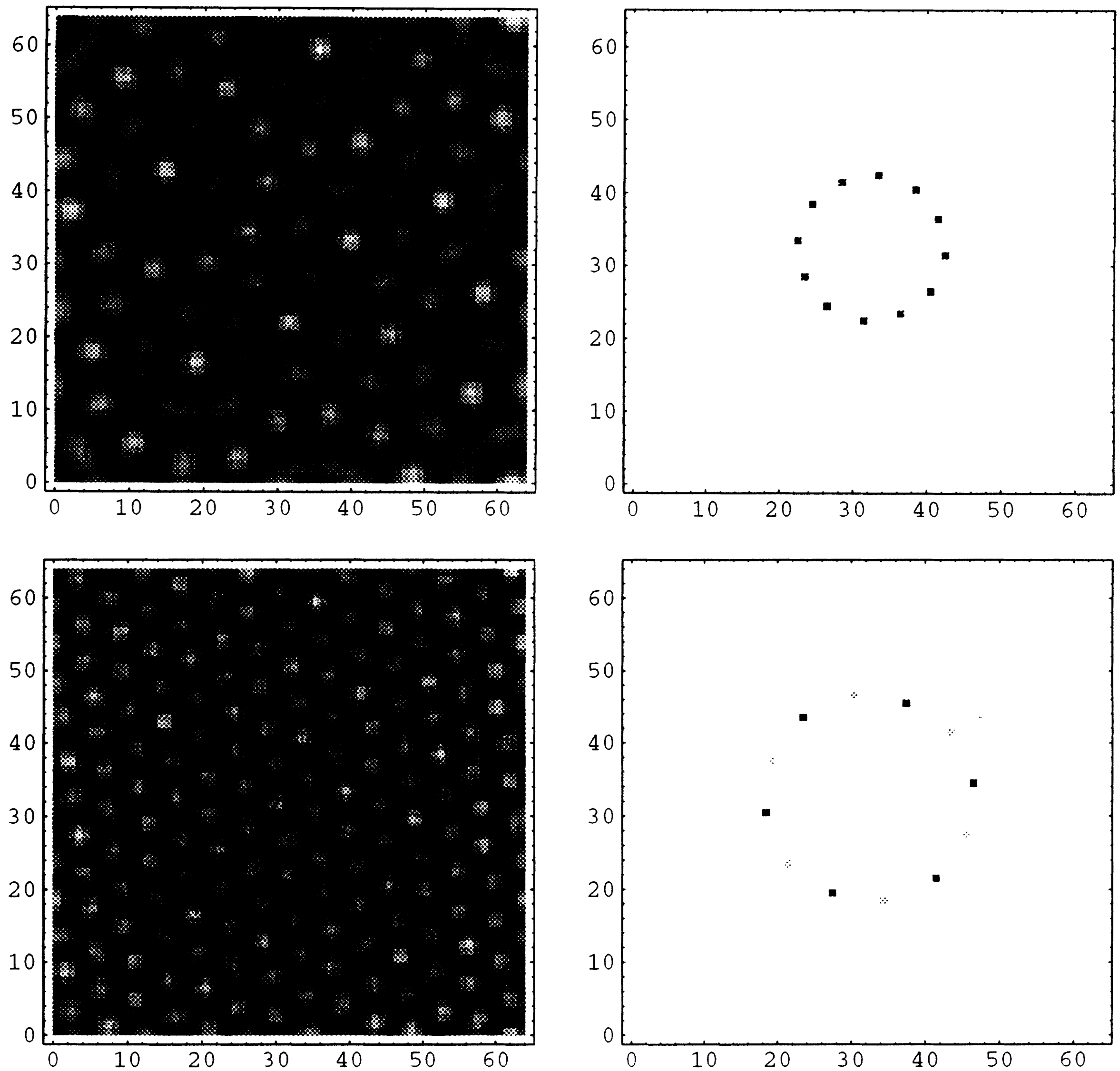

FIG. 2. Long-time solution of Eq. (10) generated numerically by a spectral code resulting from small random initial conditions $u_1(x,y), u_2(x,y) = O(10^{-3})$. On the left: shadow graphs of the fields u_1 and u_2 [axis labels denote length (x) and width (y) of the box in arbitrary units]; on the right: shadow graphs of the associated two-dimensional power spectra (axis labels denote channels of the two-dimensional FFT.)

former ones are stabilized by a pure cubic mechanism [via Eq. (2)] with an appropriate coupling function $\beta(\theta)$. Backwards bifurcating quasipatterns are selected by means of a quadratic interaction between two sets of unstable modes (k_c, q_c). A situation like this generically appears in multicritical systems close to a bicritical point. This stabilization is most effective close to the threshold, where the influence of the quadratic cross coupling prevails. The crucial point of this mechanism is the resonance condition for the triad interaction to work, which requires definite "magic" ratios k_c/q_c. This, however, is not very restrictive for systems with many (almost) critical modes (e.g., the Faraday experiment [4,5]) and/or for strongly dissipative bicritical systems [3]. In the latter case the linear growth rate of the unstable modes exhibits a broad maximum around the critical values so that wave number adjustments may take place without a considerable damping.

Discussions with W. S. Edwards, S. Fauve, T. Frisch, K. Kumar, D. Roth, and L. Tuckerman are gratefully acknowledged. This work has been supported by the Deutsche Forschungsgemeinschaft and the EEC.

[1] See, for instance, D. S. Shechtman, I. Blech, D. Gratias, and J. W. Cahn, Phys. Rev. Lett. **53**, 1951 (1984).
[2] B. Christiansen, P. Alstrom, and M. T. Levinsen, Phys. Rev. Lett. **68**, 2157 (1992).
[3] W. S. Edwards and S. Fauve, C. R. Acad. Sci. **315-II**, 417 (1992); Phys. Rev. E **47**, R788 (1993).
[4] J. B. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[5] The prefactor $\frac{1}{3}$ avoids an extra scaling of the mode amplitudes in the subsequent calculation.
[6] C. J. Chapman and M. R. E. Proctor, J. Fluid Mech. **101**, 759 (1980).
[7] A. C. Newell and Y. Pomeau (unpublished).
[8] B. A. Malomed, A. A. Nepomnyashchii, and M. I. Tribelskii, Sov. Phys. JETP **69**, 388 (1989).
[9] N. D. Mermin and S. M. Trojan, Phys. Rev. Lett. **54**, 1524 (1985).
[10] This is a generalization of the stability analysis of [8].
[11] A. C. Newell and J. A. Whitehead, J. Fluid Mech. **38**, 279 (1969).
[12] H. W. Müller, Phys. Rev. Lett. **71**, 3287 (1993).
[13] T. Frisch (private communication).

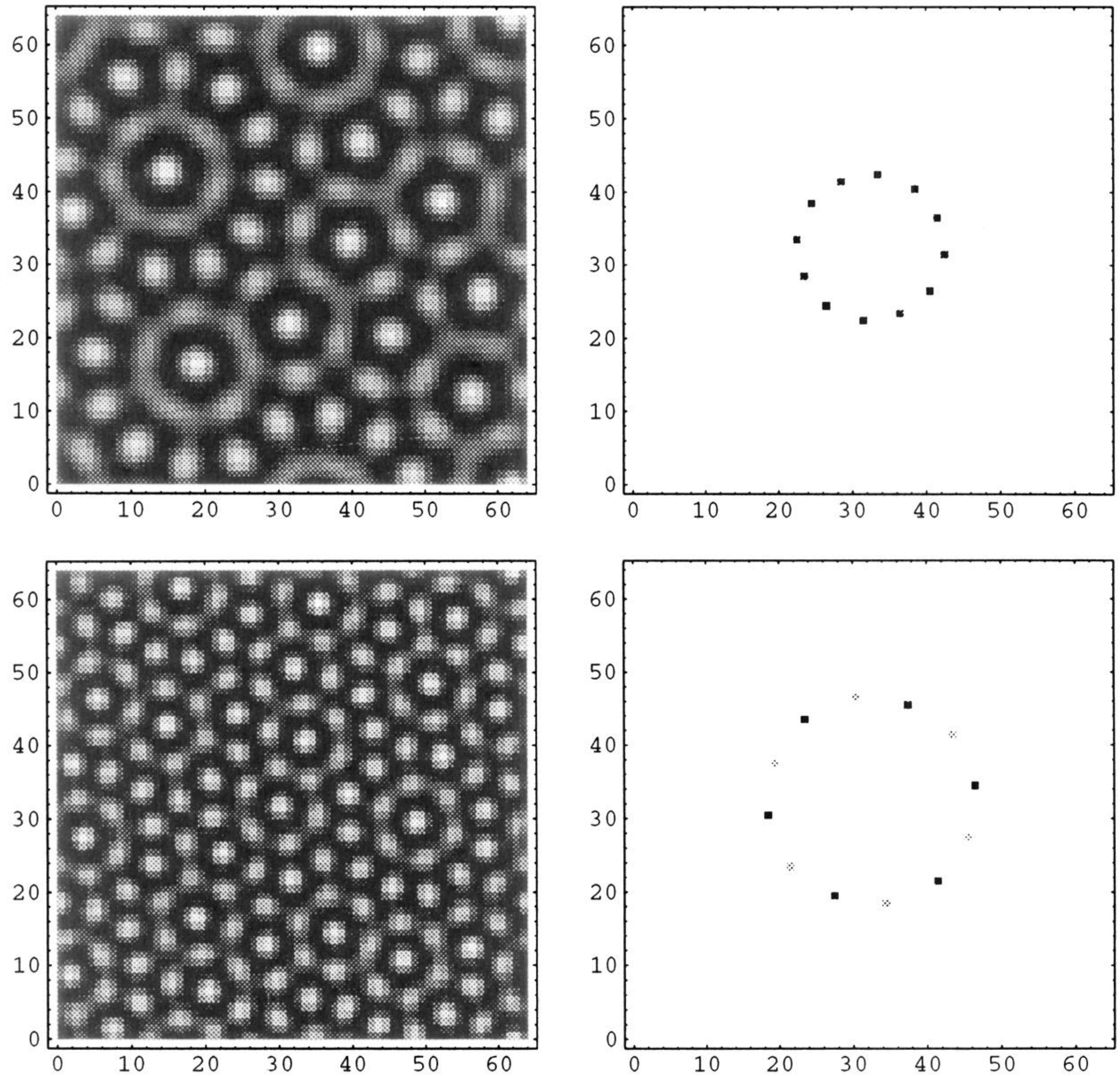

FIG. 2. Long-time solution of Eq. (10) generated numerically by a spectral code resulting from small random initial conditions $u_1(x,y), u_2(x,y) = O(10^{-3})$. On the left: shadow graphs of the fields u_1 and u_2 [axis labels denote length (x) and width (y) of the box in arbitrary units]; on the right: shadow graphs of the associated two-dimensional power spectra (axis labels denote channels of the two-dimensional FFT.)